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PL INVOLUTIONS OF FIBERED 3-MANIFOLDS

BY

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ABSTRACT. Let h be a PL involution of $F \times [0,1]$ such that $h(F \times \{0,1\}) = F \times \{0,1\}$, where F is a compact 2-manifold. It is shown that h is equivalent to an involution h' of the form $h'(x,t) = (g(x), \lambda(t))$. This result is applied to classify the PL involutions of closed, orientable, Seifert manifolds when the fixed-point set contains a two-dimensional component of negative characteristic.

This paper is concerned with the problem of classifying the PL involutions of Seifert manifolds [15] that have a two-dimensional fixed-point set. We obtain a complete classification of these involutions on closed, orientable Seifert manifolds when the fixed-point set contains a two-dimensional component of negative characteristic. Crucial to this classification, as well as of interest in its own right, is the following characterization of PL involutions on product line bundles.

THEOREM A. *Let F be a compact surface and let h be a PL involution of $F \times I$ such that $h(F \times \partial I) = F \times \partial I$ (I denotes the unit interval). Then there exists a map g of F (with g^2 = identity) such that h is equivalent to the involution h' of $F \times I$ defined by $h'(x,t) = (g(x), \lambda(t))$ for $(x,t) \in F \times I$ and $\lambda(t) = t$ or $1 - t$.*

This theorem has played a fundamental role in our study of PL involutions of 3-manifolds (see [9], [10] and [19]). An analogous result, restricted to fixed-point free involutions, was proved and used by F. Waldhausen [21]. To illustrate the use of this theorem, consider a PL involution h of a compact, P^2-irreducible 3-manifold M and suppose that there is a two-dimensional component F of Fix(h). In §2 we observe that F must be π_1-*injective* in M; that is, the homomorphism $\pi_1(F) \to \pi_1(M)$ induced by the inclusion is injective. Suppose that $M = T \times S^1$ (where T is a closed surface) and F has negative characteristic. It follows that $(T \times S) - F$ is homeomorphic to $F \times (0,1)$ and h defines an involution on $F \times (0,1)$ which can be extended to $F \times I$. Thus

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Theorem A provides a description of h. This point of view is used in the proof of the next theorem to give a complete classification of PL involutions h of closed, orientable Seifert manifolds when each component of $\mathrm{Fix}(h)$ is two-dimensional.

THEOREM B. *Let M be a closed, orientable Seifert manifold. Then M admits a PL involution h, where each component of* $\mathrm{Fix}(h)$ *is two-dimensional and at least one has negative characteristic, if and only if M is an S^1-bundle over a surface of negative characteristic that admits a cross-section. Thus, if the bundle is principle then M is a product $T \times S^1$, and if the bundle is not principle then M is the union of two copies of a twisted I-bundle identified along their boundaries by the identity. In the case the bundle is principle, M admits exactly two such involutions, up to equivalence. If the bundle is not prinicple then M admits exactly four such involutions, up to equivalence.*

If we allow $\mathrm{Fix}(h)$ to include 0-dimensional components then we have the following theorem concerning Seifert manifolds that are fibered over S^1.

THEOREM C. *Let M denote a closed, orientable Seifert manifold that is also a fiber bundle over S^1. Then M admits a PL involution h with* $\mathrm{Fix}(h)$ *containing a two-dimensional component of negative characteristic if and only if M is homeomorphic to some $F \times I/\phi$ which is obtained from $F \times I$ by identifying $(x, 0)$ with $(\phi(x), 1)$ for each $x \in F$, where ϕ is a homeomorphism of the surface F with $\phi^2 =$ identity. In this case the involution h is uniquely determined by* $\mathrm{Fix}(h)$, *up to equivalence.*

The proofs of Theorems B and C are found in §§4 and 5, together with descriptions of the involutions involved.

In [7] we show that a PL involution of a compact nonprime 3-manifold M can be split into involutions on the prime factors of M. Thus the results of the present paper can be applied to study PL involutions of the most general compact 3-manifolds.

1. **Notation.** All spaces and maps will be in the PL category. A 3-manifold M is *irreducible* if every 2-sphere in M bounds a 3-cell in M. We say that M is *P^2-irreducible* if M is irreducible and contains no two-sided projective plane P^2. An embedding of a surface F in M is *proper* if $\partial F = F \cap \partial M$. A proper embedding of the surface F in M is *two-sided* if there exists a neighborhood of F in M of the form $F \times I$ such that $F = F \times \{\frac{1}{2}\}$. A neighborhood is said to be *small* provided it is contained in a second derived neighborhood with respect to a triangulation of M in which everything under discussion is simplicial. Two surfaces F and G which are properly embedded in M are said to be *parallel* if there exists an embedding of $F \times I$ in M such that $F = F \times 0$ and $G = \overline{(\partial(F \times I) - F \times 0)}$.

Let F be a two-sided surface properly embedded in the 3-manifold M. The manifold M' obtained by *splitting* M along F is defined by the properties: $\partial M'$ contains surfaces F' and F'' which are copies of F, and identification of F' and F'' gives a natural projection $(M', F' \cup F'') \to (M, F)$ that is a relative homeomorphism. Observe that M' is homeomorphic to $\overline{(M - U)}$ where U is a regular neighborhood of F in M.

A surface F (not a 2-sphere or disk) properly embedded in a 3-manifold M is *incompressible* in M provided that whenever D is a disk in M with $D \cap F = \partial D$, then ∂D bounds a disk in F. Observe that if F is not simply-connected and is π_1-injective in M, then F is incompressible. If F is two-sided and incompressible in M, then it follows from the loop theorem [16] that F is π_1-injective.

Let h denote a simplicial involution of a triangulated 3-manifold M. Let F denote a surface properly embedded in M as a subcomplex. We shall describe an isotopy by which we move F into *h-general position* [18]. We first move F into general position with respect to $\text{Fix}(h)$. Then, using only isotopies that keep $\text{Fix}(h)$ constant, we move $F - \text{Fix}(h)$ into general position with respect to $h(F) - \text{Fix}(h)$. This is done by the usual method of shifting subcomplexes into general position. In general, $F \cap h(F)$ is a graph with the branch points of the graph lying in $\text{Fix}(h)$.

Let F denote a surface in h-general position. We assign a complexity $c(F) = (a, b)$ to F where a denotes the number of components of $F \cap h(F) - \text{Fix}(h)$ and b denotes the number of components of $\text{Fix}(h) \cap F$. We order these complexities lexicographically.

A surface E contained in $h(F)$ is said to be *innermost* if $E \cap F \subset \partial E$ and $\partial E - (E \cap F) \subset \partial M$.

2. **Two-dimensional components of** $\text{Fix}(h)$. Let h be an involution of a compact P^2-irreducible 3-manifold M. In this section we observe that each two-dimensional component of $\text{Fix}(h)$ is π_1-injective in M. This is actually an immediate corollary of Theorem 6.1 of P. Conner and F. Raymond [2] and the proof given below uses their point of view.

LEMMA 1. *Let M be a compact, P^2-irreducible 3-manifold and let h be a PL involution of M. Then every two-dimensional component of the fixed-point set of h is π_1-injective in M.*

PROOF. Choose a basepoint $x_0 \in F$ and let $p\colon (\tilde{M}, \tilde{x}_0) \to (M, x_0)$ be the universal covering space. Lift h to an involution $\tilde{h}$ of $\tilde{M}$ with $\tilde{h}(\tilde{x}_0) = \tilde{x}_0$ and let $\tilde{F} = \text{Fix}(\tilde{h})$. One easily checks that $p(\tilde{F}) = F$. It follows from Smith theory that $\tilde{F}$ is connected and simply connected since $\tilde{M}$ is either contractible or a homology 3-sphere. Hence $p|\tilde{F}\colon \tilde{F} \to F$ is the universal covering space of F. Observe that a loop in F based at x_0 is contractible in M if and only if it lifts to a loop in $\tilde{M}$ based at $\tilde{x}_0$ (and thus in $\tilde{F}$). It follows that F is π_1-injective in M.

This result can be extended to irreducible 3-manifolds which contain two-sided projective planes if we restrict our attention to the two-sided components of the fixed-point set. The proof of this next lemma will also introduce a basic construction used in the following section.

LEMMA 2. *Let M be a compact, irreducible 3-manifold and let h be a PL involution on M. Suppose that F is a two-sided, two-dimensional component of* $\mathrm{Fix}(h)$. *Then F is π_1-injective in M.*

PROOF. If $\pi_1(F)$ is trivial, then there is nothing to show. If $\pi_1(F)$ is not trivial, then it is sufficient to prove that F is incompressible in M. Let S be an arbitrary disk in M such that $\partial S = S \cap F$. We shall be finished if we can show that ∂S also bounds a disk in F.

Since h interchanges the sides of F we may assume that $(\mathring{S} \cup h(\mathring{S})) \cap \partial S = \varnothing$. Move $\mathring{S}$ into h-general position by an isotopy that keeps ∂S fixed. Among all such disks with boundary ∂S, we assume that we have chosen a disk S whose interior $\mathring{S}$ has minimal complexity.

Suppose that $\mathring{S} \cap h(\mathring{S}) \neq \varnothing$. Then there exists a simple closed curve J in $h(\mathring{S}) \cap \mathring{S}$ that bounds an innermost disk E in $h(S)$. The curve J separates S into an open disk E_1 and an annulus E_2, where $J = \bar{E}_1 \cap \bar{E}_2$. Let S' denote the disk $E \cup \bar{E}_2$. We are going to move S' with a small isotopy to ensure the h-general position of S' and to obtain $c(S') < c(S)$. For future reference, we refer to the construction of this isotopy as an *α-operation*. Let U be a small regular neighborhood of E in M such that $U \cap S$ is a regular neighborhood of J in S. Choose a disk E' close to E in U such that (i) $E' \cap S = \partial E'$; (ii) $E' \cap h(S) = J \cap \mathrm{Fix}(h)$; (iii) in E_2, $\partial E' \cup J$ bounds an annulus A pinched along $J \cap \mathrm{Fix}(h)$ (that is, A is homeomorphic to the quotient space of $J \times I$ obtained by identifying $y \times I$ to a point for each $y \in J \cap \mathrm{Fix}(h)$); (iv) the interior of the 3-cell in U bounded by $E \cup A \cup E'$ is disjoint from $S \cup h(S)$. We define S'' to be the disk $E' \cup (S' - (A \cup E))$ whose boundary is ∂S. The disk S'' may fail to intersect $\mathrm{Fix}(h)$ transversally along $J \cap \mathrm{Fix}(h)$. If this occurs, then we equivariantly push S'' away from $h(S'')$ at those points where S'' is tangent to $\mathrm{Fix}(h)$.

We have constructed a new disk S'' in h-general position such that $\partial S'' = \partial S$. Moreover, $c(\mathring{S}'') < c(\mathring{S})$ since the intersection of $\mathring{S}$ and $h(\mathring{S})$ has been simplified along J and has not been increased elsewhere during the construction of S''. Thus we must have had $c(\mathring{S}) = (0,0)$ since in the beginning $\mathring{S}$ was chosen to have minimal complexity.

Now consider the 2-sphere $\Sigma = S \cup h(S)$. This 2-sphere Σ is invariant under h and bounds a 3-cell B in M. The fixed-point set of the involution $h|B$

is precisely $F \cap B$. It follows that $F \cap B$ is a disk in F with boundary ∂S. This proves that F is incompressible in M.

3. **Involutions of** $F \times I$. The proof of Theorem A is given in this section. Let F denote a compact surface and suppose that h is a PL involution of $F \times I$ such that $h(F \times \partial I) = F \times \partial I$. Theorem A states that there is a product fibering $F \times I$ and a map g of F such that $h(x,t) = (g(x), \lambda(t))$, where $\lambda(t) = t$ or $1 - t$. We shall follow Waldhausen's point of view [21] in our proof. Let $p\colon F \times I \to F$ denote the projection onto the factor F. A subspace X of $F \times I$ is said to be *vertical* if $X = p^{-1}(p(X))$. Thus X is vertical if and only if $X = p(X) \times I$. In [21] Waldhausen proves the following lemma for compact, orientable surfaces F. However, his proof essentially goes through for any compact surface F.

LEMMA 3. *In $F \times I$, let G be such that each component of G is either a properly embedded disk which intersects $\partial F \times I$ in two vertical arcs, or an incompressible annulus which has one boundary curve in $F \times 0$ and the other in $F \times 1$. Then there exists an isotopy, constant on $(F \times 0) \cup (\partial F \times I)$, which makes G vertical.*

It follows from this lemma that we may deform the given fibering, holding $(F \times 0) \cup (\partial F \times I)$ constant, to make G vertical with respect to this new fibering, which we shall still denote by $F \times I$.

LEMMA 4. *Suppose that $\partial F \neq \varnothing$ and $h|(\partial F \times I)$ has the form $(x,t) \mapsto (g(x), \lambda(t))$. Let some component of $\partial F \times I$ not be separated by $\partial(\mathrm{Fix}(h))$. If F is not a disk, then there exists a disk D such that* (i) *∂D is not contractible in $\partial(F \times I)$,* (ii) *either $h(D) = D$ or $h(D) \cap D = \varnothing$, and* (iii) *there exists a deformation of the fibering $F \times I$, constant on $(F \times 0) \cup (\partial F \times I)$, after which $D \cup h(D)$ is vertical and $h|(D \cup h(D))$ has the form $(x, t) \mapsto (g(x), \lambda(t))$.*

PROOF. Start with a vertical disk S properly embedded in $F \times I$ such that ∂S is not contractible in $\partial(F \times I)$. Because $h|(\partial F \times I)$ preserves the fibers and $\partial(\mathrm{Fix}(h))$ does not separate $\partial F \times I$, we may require that $S \cap (\partial F \times I)$ consist of two vertical arcs k_1, k_2 such that $h(k_1 \cup k_2) \cap (k_1 \cup k_2) = \varnothing$. Move S into h-general position by an isotopy constant on $\partial F \times I$. We may assume that we have chosen S to have minimal complexity $c(S)$ among all such disks. We shall show that either $c(S) = (0,0)$ or else we can construct an invariant disk D for which ∂D is not contractible in $\partial(F \times I)$ and $D \cap (\partial F \times I)$ consists of two vertical arcs.

Suppose that $h(S) \cap S \neq \varnothing$. There exists an innermost disk E in $h(S)$ such that $J = E \cap S$ is either an arc or a simple closed curve. In either case, J separates S into two components, E_1 and E_2, such that $\overline{E}_1 \cap \overline{E}_2 = J$. If J is a simple closed curve, then one of these components, say E_1, is an open disk with boundary J. Take S' to be the disk $E \cup E_2$. We use an α-operation to move S' into h-general position and obtain a new disk S'' such that $\partial S'' = \partial S$ and $c(S'') < c(S)$. However, since this contradicts our choice of S, the curve J must be an arc.

Suppose that J is an arc with both endpoints in the same component of $F \times \partial I$, say $F \times \{i\}$. Let E_1 denote the component of $S - J$ with $\partial E_1 \subset J \cup (F \times \{i\})$. Then $E \cup E_1$ is a disk parallel to a disk in $F \times \{i\}$. Let S' denote the disk $E \cup E_2$, where $\partial S'$ is isotropic to ∂S in $\partial(F \times I)$. We shall move S' into h-general position, thus obtaining a new disk S'' with $\partial S''$ noncontractible in $\partial(F \times I)$ and $c(S'') < c(S)$. This isotopy is constructed using the following β-operation.

β-operation. Let U denote a small regular neighborhood of E in M such that $U \cap S$ is a regular neighborhood of J in S. Take a disk E' near E in U satisfying the properties: (i) $E' \cap S = L$, where

$$L = \overline{(\partial E' - (\partial (F \times I) \cap E'))}\,;$$

(ii) $E' \cap h(S) = J \cap \mathrm{Fix}(h)$; (iii) in $E_2, L \cup J$ bounds a subspace A homeomorphic to $J \times I$ pinched along $J \cap \mathrm{Fix}(h)$; and (iv) the component of $(F \times I) - (E \cup A \cup E')$ contained in U does not meet S. Let S'' denote the disk $E' \cup (S' - (A \cup E))$, which is isotopic to S'. As before during the α-operation, S'' may fail to meet $\mathrm{Fix}(h)$ transversally along some components of $J \cap \mathrm{Fix}(h)$. We use a small equivalent isotopy to move S'' and $h(S'')$ apart along these components of $J \cap \mathrm{Fix}(h)$. We thus obtain the desired disk S'' with $c(S'') < c(S)$. Again, this contradicts our choice of S.

Thus J must be an arc with one endpoint lying in each component of $F \times \partial I$. Observe that if $\partial(E \cup E_1)$ bounds a disk in $\partial(F \times I)$, then $\partial(E \cup E_2)$ is homotopic to ∂S in $\partial(F \times I)$. Thus at least one of the disks $E \cup E_1$ and $E \cup E_2$, say $S' = E \cup E_2$, has a boundary that is not contractible in $\partial(F \times I)$. If $h(S') \neq S'$, then we move S' into h-general position by a β-operation to again obtain a disk S'' that is in contradiction to our choice of S. Therefore, we can only have $h(S') = S'$ and thus S' is the disk D which we were to construct.

To complete the proof of Lemma 4 we observe that there exists a deformation of the fibering of $F \times I$, constant on $(F \times 0) \cup (\partial F \times I)$, after which $D \cup h(D)$ is vertical and $h|(D \cup h(D))$ has the form $(x,y) \mapsto (g(x), \lambda(t))$.

LEMMA 5. *Let F be a closed surface different from the 2-sphere. Then there exists a deformation of the product fibering $F \times I$ such that there exists an annulus A for which $A \cup h(A)$ is vertical, either $h(A) = A$ or $h(A) \cap A = \varnothing$, and $h|(A \cup h(A))$ has the form $(x,t) \mapsto (g(x), \lambda(t))$.*

PROOF. Let k be a noncontractible simple closed curve in F. Then $k \times I$ is an incompressible annulus in $F \times I$. Move $k \times I$ into h-general position to obtain the incompressible annulus S. Recall from Lemma 3 that there is a deformation of the fibering of $F \times I$ which makes S vertical. So if $c(S) = (0,0)$ there is nothing more to do. If $c(S) > (0,0)$, then we construct another vertical annulus S'' such that either S'' is invariant under h or $c(S'') < c(S)$ and S'' is incompressible. Repetition of this construction (a finite number of times) yields the desired vertical annulus.

Let us suppose that $h(S) \cap S \neq \varnothing$. First some observations about the components of $h(S) - (S \cap h(S))$. If $S \cap h(S)$ contains a contractible simple closed curve or an arc with both endpoints in the same component of $F \times \partial I$, then there exists an innermost disk E in $h(S)$. Since S is incompressible, the disk E is parallel to a disk in S. Also, such an innermost disk E exists in $h(S)$ whenever $S \cap h(S)$ contains, at the same time, a noncontractible simple closed curve and two arcs with endpoints in each component of $F \times \partial I$.

Case 1. There exists an innermost disk E in $h(S)$ such that $J = E \cap S$ is connected. Then E is parallel to a disk E_1 in S. The simple closed curve or arc J separates S into two components, E_1 and E_2. Let S' denote the annulus $E \cup E_2$. Observe that S' is isotopic to S. We use either an α- or a β-operation to move S' into h-general position and obtain an incompressible annulus S'' with $c(S'') < c(S)$.

Case 2. Case 1 does not occur and there exists an innermost disk E in $h(S)$ such that $J = E \cap S$ consists of two arcs, J_1 and J_2. Each of the arcs J_i has an endpoint in both components of $F \times \partial I$. Since at least one of the two annulli $E \cup E_1$ and $E \cup E_2$ is incompressible, we lose no generality by assuming that $S' = E \cup E_2$ is incompressible in $F \times I$. If $h(E) = E_2$, then we take S'' to be the invariant annulus S'. On the other hand, if $h(E) \neq E_2$, we have three possibilities that arise. Let U be a small regular neighborhood of E such that $U \cap S$ is a regular neighborhood of J in S. The disk $U \cap h(S)$ is, of course, two-sided in U. It is possible for $E_2 \cap (U - (U \cap h(S)))$ to meet either only one or both components of $U - (U \cap h(S))$. In the first two subcases we distinguish between these two possibilities. In the third subcase it makes no difference. Remember that in the following subcases we are working under the assumption that $h(E) \neq E_2$.

Subcase (a). The disk E_2 meets both components of $U - (U \cap h(S))$ and either $h(J) = J$ or $h(J) \cap J = \varnothing$. Choose a disk E' near E in U such that (i) $E' \cap E = (J \cap \operatorname{Fix}(h)) \cup K$, where K is an arc in $E - J$ parallel to J_1 and E' crosses E at K, (ii) $E' \cap (S \cup h(S)) = K \cup L$, where L denotes the two arcs $\overline{(\partial E' - (F \times \partial I)}$, (iii) $L \cup J$ bounds the set A in E_2, where A corresponds to the quotient space obtained from $J \times I$ by identifying $y \times I$ to a point for each $y \in J \cap \operatorname{Fix}(h)$, and (iv) the two components of $F \times I - (E \cup A \cup E')$ contained in U are disjoint from $S \cup h(S)$. We define $S'' = E \cup (E_2 - A)$.

By the usual method, equivariantly move S'' off any points of $J \cap \text{Fix}(h)$ at which S'' is tangent to $\text{Fix}(h)$ to obtain S'' in h-general position and $c(S'') < c(S)$.

Subcase (b). The disk E_2 meets only one component of $(U - (U \cap h(S)))$ and either $h(J) = J$ or $h(J) \cap J = \varnothing$. We obtain S'' by moving S' into h-general position by an isotopy similar to that defined by a β-operation. The same description applies if we allow L and J each to be two arcs instead of one in the β-operation.

Subcase (c). $h(J) \cap J = J_1$, a single arc. Choose a disk E' near E in U such that (i) $E \cap E' = J_1 \cup (J_2 \cap \text{Fix}(h))$; (ii) $E' \cap (S \cup h(S)) = E' \cap S = J_1 \cup L$, where $L = \overline{(\partial E' - (J_1 \cup F \times \partial I))}$; (iii) $L \cup J_2$ bounds A in E_2, where A is homeomorphic to the quotient space obtained from $J_2 \times I$ by identifying $y \times I$ to a point for each $y \in J_2 \cap \text{Fix}(h)$; and (iv) the component of $F \times I - (E \cup A \cup E')$ contained in U is disjoint from $S \cup h(S)$. Define S'' to be the annulus $E' \cup (E_2 - A)$. As usual, move S'' into h-general position near $J \cap \text{Fix}(h)$ to obtain $c(S'') < c(S)$.

Case 3. There does not exist an innermost disk in $h(S)$ and $S \cap h(S) \cap \partial(F \times I) = \varnothing$. In this case there exists an innermost annulus E in $h(S)$ such that ∂E has one component in common with $\partial(h(S))$. Let J denote the noncontractible, simple closed curve $E \cap S$. Consider a small regular neighborhood U of E such that $U \cap S$ is a regular neighborhood of J in S. Let E_1 and E_2 denote the components of $S - J$, labeled such that $\partial E_1 - J$ and $\partial E - J$ lie in the same component of $F \times \partial I$. We define S' to be the annulus $E \cup E_2$. If $h(S') = S'$, then we take $S'' = S'$. If S' is not invariant then we use the following isotopy to move S' into h-general position.

Let E' denote an annulus near E in U such that (i) $E' \cap h(S) = J \cap \text{Fix}(h)$; (ii) $\partial E' \cup \partial E$ bounds an annulus A in $F \times \partial I$ and an annulus A' in E_2 that is pinched along $J \cap \text{Fix}(h)$; (iii) $S \cup h(S)$ does not meet the interior of the component of $F \times I - (E \cup A \cup A' \cup E')$ that is contained in U. We define S' to be $E' \cup (S' - (A' \cup E))$. Now move S'' into h-general position near $J \cap \text{Fix}(h)$ to obtain $c(S'') < c(S)$.

Case 4. $S \cap h(S)$ contains an invariant arc J that has one endpoint in each component of $F \times \partial I$. Take a small, invariant regular neighborhood U of J which we may assume is vertical since J is contained in the vertical annulus S. Take S'' to be the vertical annulus ∂U.

Observe in Cases 1–3 that the existence of a deformation of the fibering which makes S'' vertical follows from Lemma 3.

We complete the proof of Lemma 5 by changing the parametrizations of the fibers to obtain $h|S'' \cup h(S'')$ in the form $(x, t) \mapsto (g(x), \lambda(t))$ and taking $A = S''$.

Proof of Theorem A. We consider first the case when $\partial F \neq \varnothing$. In view of the assumption $h(F\times\partial I) = F\times\partial I$, we may suppose that $h|\partial F\times I$ already has the form $(x,t)\mapsto(g(x),\lambda(t))$. We shall use induction on the characteristic $\chi(F)$ of F to prove that there exists a deformation of the fibering, constant on $\partial F\times I$, after which h will have the desired form.

In the case when F is a disk the theorem is well known ([11], [12], and [22]). Suppose that $\chi(F) < 1$. If $\partial(\mathrm{Fix}(h))$ does not separate $\partial F\times I$, then Lemma 4 is applicable. It follows that there is a disk D such that ∂D is not contractible in $\partial(F\times I)$, either $h(D) = D$ or $h(D)\cap D = \varnothing$, and there is a deformation of the fibering, constant on $(F\times 0)\cup(\partial F\times I)$, after which $D\cup h(D)$ is vertical and $h|D\cup h(D)$ has the desired form. Split $F\times I$ along $D\cup h(D)$ to obtain $F'\times I$ and an involution h'. Apply the induction hypothesis to obtain a deformation of the fibering $F'\times I$, constant on $(F'\times 0)\cup(\partial F'\times I)$, after which $h'(x,t) = (g(x),\lambda(t))$. Repairing the cut along $D\cup h(D)$ defines the required fibering of $F\times I$.

If $\partial(\mathrm{Fix}(h))$ does separate $\partial F\times I$, we shall show that $\mathrm{Fix}(h)$ is a two-dimensional, two-sided surface that is isotopic to $F\times\{\frac{1}{2}\}$. First notice that each component of $\partial F\times I$ contains exactly one component of $\partial(\mathrm{Fix}(h))$ and $\mathrm{Fix}(h)\cap(F\times\partial I) = \varnothing$. Let G denote a two-dimensional component of $\mathrm{Fix}(h)$. Since G is disjoint from $F\times\partial I$, it follows that G must be two-sided. Now observe that either $(F\times I) - G$ is connected or else G separates $F\times I$ into two components, each containing a component of $F\times\partial I$. However, it is not possible for $(F\times I) - G$ to be connected since the homomorphism $\pi_1(F\times 0)\to\pi_1(F\times I)$ induced by inclusion is an isomorphism. Therefore it follows from [1] that G is isotopic to $F\times\{\frac{1}{2}\}$. Since ∂G is already equal to $\partial F\times\{\frac{1}{2}\}$, there is a deformation of the fibering, constant on $\partial F\times I$, after which $G = F\times\{\frac{1}{2}\}$. Now it is easy to redefine the fibering to obtain $h(x,t) = (g(x), 1-t)$, for $(x,t)\in F\times I$. This completes the proof of Theorem B for the case when $\partial F\neq\varnothing$.

Now suppose that $\partial F = \varnothing$. The case when F is a 2-sphere is again well known ([11], [12], and [22]). Thus suppose that $\chi(F) < 2$. We use Lemma 5 to find an annulus A and a deformation of the fibering $F\times I$ such that $A\cup h(A)$ is vertical, either $h(A) = A$ or $h(A)\cap A = \varnothing$, and $h|(A\cup h(A))$ has the form $(x,t)\mapsto(g(x),\lambda(t))$. Split $F\times I$ along $A\cup h(A)$ to obtain $F'\times I$ and an involution h'. Since $\partial F'\neq\varnothing$, we may find a deformation of the fibering $F'\times I$, constant on $\partial F'\times I$, after which h' has the form $(x,t)\mapsto(g(x),\lambda(t))$. Repairing the cut along $A\cup h(A)$ gives the required fibering of $F\times I$. This completes the proof of Theorem A.

4. Seifert manifolds fibered over S^1. In this section we consider PL involutions h on closed, orientable Seifert manfiolds that are fiber bundles over S^1. We assume that $\mathrm{Fix}(h)$ contains a two-dimensional component F with $\chi(F) < 0$. Before beginning the proof of Theorem C we shall discuss some facts concerning these Seifert manifolds that contain a π_1-injective surface of negative characteristic.

It follows from Theorem 4 of [14] that $\pi_1(M)$ has a nontrivial center, which we denote by $\rho\pi_1(M)$. Suppose that G is a one-sided surface in M. Consider a regular neighborhood U of G in M. Then ∂U is a two-sided surface that separates M. We point out that G is π_1-injective in M if and only if ∂U is. Hence, when G is π_1-injective we can view $\pi_1(M)$ as a generalized free product amalgamated along the subgroup $\pi_1(\partial U)$. This implies that $\rho\pi_1(M)$ is contained in $\rho\pi_1(\partial U)$, which is trivial unless $\chi(G) = 0$. Therefore any π_1-injective surface G of negative characteristic in M is necessarily two-sided and nonseparating. If we split M along such a π_1-injective surface G we obtain $G \times I$ [20]. From this we obtain $M = G \times I/\phi$ where ϕ is the homeomorphism of G repairing this cut. The fact that $\rho\pi_1(M)$ is nontrivial implies that ϕ is isotopic to a periodic homeomorphism of G [14]. It then follows that the center of $\pi_1(M)$ is infinite cyclic and has only the identity element in common with the subgroup $\pi_1(G)$.

Proof of Theorem C. Let us view M as $T \times I/\phi$ where T has the maximal characteristic among all possible fiberings of M over S^1. Let F denote a two-dimensional component of Fix(h) such that $\chi(F) < 0$. We have just seen that $M - F$ is homeomorphic to $F \times (0,1)$. Any remaining two-dimensional components of Fix(h) are π_1-injectively embedded in $F \times (0,1)$ and thus also have negative characteristic. Suppose that F' is a second two-dimensional component of Fix(h). Since F is π_1-injectively embedded in $F' \times (0,1)$, we see that $\pi_1(F)$ and $\pi_1(F')$ are isomorphic to subgroups of each other. It follows from the fact that F and F' have negative characteristics that the homomorphism induced on the fundamental groups by the inclusion $F' \subset F \times (0,1)$ is an isomorphism. We may apply Theorem 7.2 of [1] to deform the fibering of $F \times (0,1)$ to obtain $F' = F \times \{\frac{1}{2}\}$. Since h interchanges the sides of F' it is clear that Fix(h) can have at most two components of dimension two.

Case 1. Fix(h) has two 2-dimensional components, F and F'. As we have just seen, $F \cup F'$ splits M into two components, say X and Y, such that $h(X) = Y$. Both $\overline{X}$ and $\overline{Y}$ are homeomorphic to $F \times I$. We define a retraction $r: M \to \overline{X}$ by setting $r(x) = x$ if $x \in \overline{X}$ and $r(x) = h(x)$ if $x \in \overline{Y}$. This retraction induces an epimorphism $r_*: H_1(M) \to H_1(\overline{X})$. Applying the Van Kampen theorem to M, viewed as $T \times I/\phi$, we obtain another epimorphism $g: H_1(T) \oplus H_1(S^1) \to H_1(M)$. Notice that $r_* \circ g(H_1(S^1)) = 0$. Hence $r_* \circ g$ defines an epimorphism of $H_1(T)$ onto $H_1(F)$. This implies that $\chi(T) \leqslant \chi(F)$. Since T was chosen to be a fiber of maximal characteristic, we conclude that F is homeomorphic to T.

Let us choose a fixed product structure $F \times [0, \frac{1}{2}]$ for $\overline{X}$. We use h and this preferred fibering of $\overline{X}$ to define a fibering $F \times [\frac{1}{2}, 1]$ of $\overline{Y}$ such that $h(x,t)$

$= (x, 1-t)$ for $(x,t) \in F \times [0,\frac{1}{2}]$. By sewing $\overline{X}$ and $\overline{Y}$ together along $F \cup F'$, we obtain a product structure $F \times S^1$ for M such that $h(x,t) = (x, 1-t)$.

Case 2. Fix(h) contains only one 2-dimensional component F. Split M along F to obtain $F \times I$. The involution h defines an involution h' of $F \times I$ such that $h'(F \times 0) = F \times 1$. It follows from Theorem A that the product structure $F \times I$ can be chosen such that $h'(x,t) = (g(x), 1-t)$, where g is an involution of F. Repairing the cut along F, we may view M as $F \times I/g$ and $h([x,t]) = [g(x), 1-t]$. Since h reserves the orientation of M, the involution g is orientation preserving and has a discrete fixed-point set or $\mathrm{Fix}(g) = \varnothing$.

We require the orientability of F to prove the uniqueness of this involution h having F as the only two-dimensional component of Fix(h). Suppose that g and g' are both orientation preserving involutions of F and M is homeomorphic to both $F \times I/g$ and $F \times I/g'$. Let h and h' denote the involutions defined on $F \times I/g$ and $F \times I/g'$, respectively, by $[x, t] \mapsto [g(x), 1-t]$ and $[x, t] \mapsto [g'(x), 1-t]$. Heil observes in [3] that there exists a homeomorphism k of F such that $g = kg'k^{-1}$. Then a homeomorphism K: $F \times I/g' \to F \times I/g$ is defined by $[x, t] \mapsto [k(x), t]$ with the property that $h = Kh'K^{-1}$. This completes the proof of Theorem C.

The proof for that part of Theorem B concerning fiber bundles over S^1 follows from Theorem C and the next lemma. Recall that in Theorem B we assume there are no 0-dimensional components of Fix(h).

LEMMA 6 [17]. *Let F be a closed orientable surface and suppose that $\chi(F) < 0$. If g is a fixed-point free, orientation preserving involution of F with orbit space T, then $F \times I/g$ is homeomorphic to $T \times S^1$.*

5. **Twisted I-bundles.** In this section we complete the proof of Theorem B and in the process treat involutions of twisted I-bundles. For instance, we show that an orientable, twisted I-bundle over a closed surface with even negative characteristic admits exactly four distinct free involutions. We first consider twisted I-bundles and then move on to consider Seifert manifolds which are the union of two twisted I-bundles.

Let α be a free involution of the closed surface S. We let $M(\alpha)$ denote the twisted I-bundle over S/α obtained from $S \times I$ by identifying $(x, 1)$ with $(\alpha(x), 1)$ for each $x \in S$. The points of $M(\alpha)$ are denoted by $[x, t]$ for $(x, t) \in S \times I$.

Suppose we have two free involutions α and α' of S with homeomorphic orbit spaces S/α and S/α'. Let p: $S \to S/\alpha$ and p': $S \to S/\alpha'$ denote the projections. The two involutions α and α' are equivalent if and only if there exists an isomorphism $\pi_1(S/\alpha) \to \pi_1(S/\alpha')$ that carries $p_*(\pi_1(S))$ onto $p'_*(\pi_1(S))$. Thus the equivalence classes of free involutions on a surface S are determined by the orbit spaces and the subgroups of index two in the fundamental group of the orbit space, modulo automorphisms.

Let G denote the fundamental group of the closed surface S. Consider two subgroups H_1, H_2 of index two in G. We shall write $H_1 \sim H_2$ if there exists an automorphism of G carrying H_1 onto H_2. The subgroups of index two in G correspond to the kernels of epimorphisms $G \to Z_2$. We denote the kernel of an epimorphism $\theta: G \to Z_2$ by $K(\theta)$.

The group G may be presented by $(u_1, \ldots, u_n: r)$, where $r = [u_1, u_2] \cdots [u_{2m-1}, u_{2m}]$ and $2m = n$, or $r = u_1^2 u_2^2 \cdots u_n^2$. We define the epimorphisms $\theta(\varepsilon): G \to Z_2$ (for $\varepsilon = 0, n_2, n_3, n_4$) by specifying their action on the given generators of G:

For $r = [u_1, u_2] \cdots [u_{2m-1}, u_{2m}]$,

$$\theta(0): u_i \mapsto -1, \quad \text{for all } i;$$

For $r = u_1^2 \cdots u_n^2$,

$$\theta(n_2): u_i \mapsto -1, \quad \text{for all } i,$$

$$\theta(n_3): u_i \mapsto -1, \quad \text{for } i = 1, \ldots, n-1, u_n \mapsto 1 \ (n \geqslant 2),$$

$$\theta(n_4): u_i \mapsto -1, \quad \text{for } i = 1, \ldots, n-2, u_{n-1} \mapsto 1 \text{ and } u_n \mapsto 1 \ (n \geqslant 3).$$

Lemma 7 [13]. *Let H be a subgroup of index two in G. Then $H \sim K(\theta(\varepsilon))$ for exactly one value of ε among* $0, n_2, n_3, n_4$.

Let $\alpha(\varepsilon)$ denote the nontrivial covering transformation of the two-sheeted covering space B of the surface S determined by the subgroup $K(\theta(\varepsilon))$ of $\pi_1(S)$. The covering space B is orientable for $\varepsilon = 0, n_2$ and nonorientable for $\varepsilon = n_3, n_4$.

Lemma 8. *The following statements are equivalent*: (i) $M(\alpha(\varepsilon))$ *is homeomorphic to* $M(\alpha(\varepsilon'))$, (ii) $\varepsilon = \varepsilon'$, (iii) $\alpha(\varepsilon)$ *is equivalent to* $\alpha(\varepsilon')$.

Proof. The equivalence of (ii) and (iii) follows from Lemma 7. It is obvious that (ii) implies (i). Suppose that we have a homeomorphism $g: M(\alpha(\varepsilon)) \to M(\alpha(\varepsilon'))$. Then $g_*: \pi_1(M(\alpha(\varepsilon))) \to \pi_1(M(\alpha(\varepsilon')))$ is an isomorphism that carries $K(\theta(\varepsilon))$ onto $K(\theta(\varepsilon'))$. Therefore $\alpha(\varepsilon)$ is equivalent to $\alpha(\varepsilon')$.

Lemma 9. *Let M be a twisted I-bundle over a closed surface B. Suppose that h is a PL involution of M with a one-sided surface F for* Fix(h). *Let $\alpha = h|\partial M$. Then M is homeomorphic to $M(\alpha)$ and h is equivalent to the involution on $M(\alpha)$ defined by $[x, t] \mapsto [\alpha(x), t]$. Moreover, h is unique up to equivalence.*

Proof. If B is a projective plane then this follows from [8]. Suppose that B is not a projective plane and let $p: M \to M/h$ be the projection onto the orbit space of h. Then M/h is a compact, P^2-irreducible 3-manifold with two boundary components. The homomorphism $\pi_1(p(\partial M)) \to \pi_1(M/h)$ induced by inclusion is an isomorphism. Therefore M/h is homeomorphic to $p(\partial M) \times I$ [3]. We lift this fibering of M/h to obtain a fibering of M as a twisted I-bundle with fibers $p^{-1}(x \times I)$. We are viewing M as $M(\alpha)$, where $\alpha = h|\partial M$. It is easy to see that h sends $[x, t]$ to $[\alpha(x), t]$.

For the uniqueness, suppose that $M(\alpha)$ is homeomorphic to $M(\alpha')$ and we have an involution h' defined on $M(\alpha')$ by $h'([x,t]) = [\alpha'(x), t]$. We see by Lemma 8 that there is a map k of ∂M such that $k\alpha k^{-1} = \alpha'$. Define the homeomorphism g: $M(\alpha) \to M(\alpha')$ by $g([x,t]) = [k(x), t]$. It follows that $h' = ghg^{-1}$.

Now consider an orientation-reversing, free involution β of the closed orientable surface T_g (of odd genus g). Let B denote the orbit space of β. We defined for Lemma 8 the involutions $\alpha(\varepsilon)$ of B (for $\varepsilon = n_3$ and n_4) with orbit space S. Observe that the composition of covering space projections $T_g \to B \to S$ defines a regular covering space. Thus each $\alpha(\varepsilon)$ lifts to a unique orientation-preserving involution $\tilde{\alpha}(\varepsilon)$ of T_g which commutes with β.

LEMMA 10. *If h is a free involution, twisted I-bundle $M = M(\beta)$, then h is equivalent to one of the involutions $h(\varepsilon)$ or $\bar{h}(\varepsilon)$ defined on $M(\beta)$ as follows*:

(i) $h(\varepsilon)$: $[x,t] \mapsto [\tilde{\alpha}(\varepsilon)(x), t]$, *if h is orientation-preserving*;

(ii) $\bar{h}(\varepsilon)$: $[x, t] \mapsto [\tilde{\alpha}(\varepsilon)\beta(x), t]$, *if h is orientation-reversing*; *where $\varepsilon = n_3$ or n_4*.

For odd $g > 1$ it follows that $M(\beta)$ admits exactly two distinct orientation-preserving, free involutions and two distinct orientation-reversing, free involutions, up to equivalence. For given $g > 1$, the 3-manifold $M(\beta)$ does not admit any free involution.

PROOF. If ∂M is a 2-sphere the conclusion follows from [6] and [8], in which case there is no free involution. Thus we may assume that ∂M is not a 2-sphere, hence M/h is P^2-irreducible. Let p: $M \to M/h$ be the projection onto the orbit space of h. Define $\alpha' = h|\partial M$. The subgroup $\pi_1(\partial(M/h))$ has index two in $\pi_1(M/h)$ and so M/h is a twisted I-bundle [21]. We lift a fibering of M/h to define a fibering $M(\beta')$ of M, where β' is an orientation-reversing, free involution of the closed surface ∂M and $\beta'\alpha' = \alpha'\beta'$. If we view M with this fibering we have $h([x,t]) = [\alpha'(x), t]$. There exists a map k of ∂M such that $\beta' = k\beta k^{-1}$. Let us define the map g: $M(\beta) \to M(\beta')$ by $g([x,t]) = [k(x), t]$. Then we have the involution $h' = g^{-1}hg$ defined on $M(\beta)$ by $h'([x,t]) = [\alpha(x), t]$, where $\alpha = k^{-1}\alpha' k = h'|\partial M$.

Observe that α covers a free involution α^* of $B = T_g/\beta$. Hence $\chi(B)$ is even, from which it follows that g is odd. There exists a homeomorphism μ^* of B and an ε such that $\mu^*\alpha^*\mu^{-1} = \alpha(\varepsilon)$. The homeomorphism μ^* lifts to a homeomorphism μ of T_g such that $\mu\beta = \beta\mu$ and $\mu\alpha\mu^{-1}\tilde{\alpha}(\varepsilon)$ is equal to either 1 or β. Thus we have either $\mu\alpha\mu^{-1} = \tilde{\alpha}(\varepsilon)$ or $\mu\alpha\mu^{-1} = \tilde{\alpha}(\varepsilon)\beta$.

Let us define the homeomorphism g: $M(\beta) \to M(\beta)$ by $g([x, t]) = [\mu(x), t]$. It follows that h is equivalent to the involution $g^{-1}h'g$ of $M(\beta)$ which sends $[x, t]$ to either $[\tilde{\alpha}(\varepsilon)(x), t]$ or $[\tilde{\alpha}(\varepsilon)\beta(x), t]$.

To complete the proof of the lemma we must show that if the involutions h_1, h_2 of $M(\beta)$ defined by $[x, t] \to [\tilde{\alpha}(\varepsilon_i)(x), t]$ for $i = 1, 2$, respectively, are equivalent then $\varepsilon_1 = \varepsilon_2$. Let p_i: $M(\beta) \to M(\beta)/h_i$ be the projection onto the orbit space, for $i = 1, 2$. Suppose that g is a homeomorphism of $M(\beta)$ for which $gh_1g^{-1} = h_2$. Then g covers a homeomorphism $\bar{g}$: $M(\beta)/h_1 \to M(\beta)/h_2$ such that $\bar{g}_*(K(\theta(\varepsilon_1))) = K(\theta(\varepsilon_2))$. It follows from Lemma 7 that this implies $\varepsilon_1 = \varepsilon_2$. A similar argument holds for the orientation-reversing case.

We are ready to consider Seifert manifolds which are the union of two twisted I-bundles. Let α, β be two free, orientation-reversing involutions of the closed, orientable surface T. We shall let $M(\alpha, \beta)$ denote the space obtained from $T \times I$ by identifying $(x, 0)$ to $(\alpha(x), 0)$ and $(x, 1)$ to $(\beta(x), 1)$ for each $x \in T$. Notice that $M(\alpha, \beta)$ is homeomorphic to the space obtained by identifying $M(\alpha)$ and $M(\beta)$ along their boundaries by the identity map.

Let β be a preferred, orientation-reversing, free involution of the closed orientable surface T_g of genus g ($g > 1$). We can choose a corresponding, two-sheeted covering space p: $T_{2g-1} \to T_g$ such that there exists a free, orientation-reversing involution $\tilde{\beta}$ of T_{2g-1} covering β. Let S denote the orbit space of $\tilde{\beta}$. Consider one of the free involution $\alpha(\varepsilon)$ of S from Lemma 8, where $\varepsilon = n_3$ or n_4. The orbit space B of $\alpha(\varepsilon)$ is homeomorphic to the orbit space of β. We can lift $\alpha(\beta)$ to a unique, orientation-reversing, free involution $\tilde{\alpha}(\varepsilon)$ of T_{2g-1} such that $\tilde{\alpha}(\varepsilon)$ commutes with $\tilde{\beta}$ and $\tilde{\alpha}(\varepsilon)\tilde{\beta}$ is also fixed-point free.

The next lemma establishes some various representations of the 3-manifold $M(\beta, \beta)$.

LEMMA 11. *The closed, orientable* 3-*manifolds* $M(\beta, \beta)$ *and* $M(\tilde{\alpha}(\varepsilon), \tilde{\beta})$ *are homeomorphic.*

PROOF. Both $M(\beta, \beta)$ and $M(\tilde{\alpha}(\varepsilon), \tilde{\beta})$ are nonprinciple, orientable S^1-bundles over the nonorientable surface $B = T_g/\beta$. It follows from [14] that such nonprinciple S^1-bundles are homeomorphic to $M_b = \{b\colon (n_2, g', 0, 0)\}$, where g' is the genus of B and $b = 0$ or 1. Here $b = 0$ if and only if there exists a cross-section for the bundle. It is easy to see that the bundle $M(\beta, \beta)$ admits a cross-section. However, it is not as apparent that a cross-section can be found for the bundle $M(\tilde{\alpha}(\varepsilon), \tilde{\beta})$. We shall take an indirect approach.

From [14] we see that the torsion of the first integral homologies of M_0 and M_1 are given by $\mathrm{Tor}(H_1(M_0)) \cong Z_2 + Z_2$ and $\mathrm{Tor}(H_1(M_1)) \cong Z_4$. Thus it will be sufficient to show that $\mathrm{Tor}(H_1(M(\tilde{\alpha}(\varepsilon), \tilde{\beta})))$ is not Z_4. Of course we may compute this directly by means of the Van Kampen theorem. This fact may also be seen by considering the Mayer-Vietoris sequence for the pair $(M(\tilde{\alpha}(\varepsilon)), M(\tilde{\beta}))$ of subspaces of $M(\tilde{\alpha}(\varepsilon), \tilde{\beta})$. We have the exact sequence

$$H_1(T) \xrightarrow{(i_*, -j_*)} H_1(M(\tilde{\alpha}(\varepsilon))) \oplus H_1(M(\tilde{\beta})) \to H_1(M(\tilde{\alpha}(\varepsilon), \tilde{\beta})) \to 0.$$

A torsion element of order four in $H_1(M(\tilde{\alpha}(\varepsilon), \tilde{\beta}))$ could only occur if either i_* or j_* acted like multiplication by four on an infinite cyclic factor. However, one can see that neither i_* nor j_* acts this way by considering the Mayer-Vietoris sequences for the pairs $(M(\tilde{\beta}), M(\tilde{\beta}))$ and $(M(\tilde{\alpha}(\varepsilon)), M(\tilde{\alpha}(\varepsilon)))$ of $M(\tilde{\beta}, \tilde{\beta})$ and $M(\tilde{\alpha}(\varepsilon), \tilde{\alpha}(\varepsilon))$, respectively. Here the first homomorphisms of the corresponding sequences are given by $(i_*, -i_*)$ and $(j_*, -j_*)$, respectively. Since both of these S^1-bundles concerned admit cross-sections, the torsion part of their first homologies is $Z_2 \oplus Z_2$. It follows that both $M(\beta, \beta)$ and $M(\tilde{\alpha}(\varepsilon), \tilde{\beta})$ are homeomorphic to M_0.

The following theorem completes the proof of Theorem B.

THEOREM D. *Let M denote a closed, orientable Seifert manifold that cannot be fibered over S^1. Suppose that h is an involution of M such that each component of* Fix(h) *is two-dimensional and at least one has negative characteristic. Then there exist the preferred involutions β, $\tilde{\beta}$ and $\tilde{\alpha}(\varepsilon)$ from Lemma* 11 *such that $M \approx M(\beta, \beta) \approx M(\tilde{\alpha}(\varepsilon), \tilde{\beta})$ and h is equivalent to one of the following involutions*:

(i) $[x, t] \mapsto [x, 1 - t]$ *on* $M(\beta, \beta)$,

(ii) $[x, t] \mapsto [\beta(x), t]$ *on* $M(\beta, \beta)$,

(iii) $[x, t] \mapsto [\tilde{\alpha}(\varepsilon)(x), t]$ *on* $M(\tilde{\alpha}(\varepsilon), \tilde{\beta})$, *where* $\varepsilon = n_3, n_4$.

These four involutions are distinct.

PROOF. Let F be a component of Fix(h) with $\chi(F) < 0$.

Case 1. F is two-sided. The surface F splits M into two twisted I-bundles, M' and M''. We may view M' as $M(\beta)$. By using h to transfer this fibering of M' to M'', we may fiber M'' as $M(\beta)$ such that $h([x, t]) = [x, 1 - t]$. This defines a fibering of M as $M(\beta, \beta)$.

Suppose that h' is another involution with Fix(h') $= F'$, a two-sided surface. Then we may view M as $M(\beta', \beta')$ with $h'([x, t]) = [x, 1 - t]$. Since F/β and F'/β' are the decomposition spaces of Seifert fiberings of M, we may conclude that they are homeomorphic [14] and, hence, F is homeomorphic to F'. It follows that β and β' are conjugate. Let k: $F \to F'$ be a homeomorphism such that $\beta = k\beta'k^{-1}$. Define the homeomorphism g: $M(\beta, \beta) \to M(\beta', \beta')$ by $g([x, t]) = [k(x), t]$. Then $ghg^{-1} = h'$. Thus, up to equivalence, M admits a unique involution with Fix(h) containing a two-sided component.

Case 2. Fix(h) $= F \cup F'$, two one-sided components. Let U denote an invariant regular neighborhood of F. Then ∂U separates M into the two twisted I-bundles M' and M''. Let $h|\partial U = \beta$. It follows from Lemma 9 that we may view both M' and M'' as $M(\beta)$. Hence we may view M as $M(\beta, \beta)$ and $h([x, t]) = [\beta(x), t]$.

Suppose that h' is another such involution on M, where we view M as $M(\beta', \beta')$ and $h'([x, t]) = [\beta'(x), t]$. As in Case 1, there exists a homeomorphism k such that $\beta = k\beta' k^{-1}$. Define the homeomorphism g: $M(\beta, \beta) \to M(\beta', \beta')$ by $g([x, t]) = [k(x), t]$. Then $ghg^{-1} = h'$. Thus M admits a unique involution with two one-sided surfaces for Fix(h).

Case 3. Fix(h) $= F$, a one-sided surface. Let U denote an invariant regular neighborhood of F with $T = \partial U$. Let $\rho = h|T$. It follows from Lemmas 9 and 10 that we may fiber M as $M(\rho, \mu)$ such that h corresponds to $[x, t] \to [\rho(x), t]$ and μ is a free involution of T. Then ρ covers a free involution ρ^* of T/μ. Let B denote the orbit space of ρ^*. Clearly M is a nonprinciple S^1-bundle over B and is homeomorphic to $M(\beta, \beta)$ for a free orientation-reversing involution β of T_g. Let us follow the notation introduced by Lemma 11. Then $\tilde{\beta}$ denotes an orientation-reversing involution of $T = T_{2g-1}$ covering the involution β. Since μ is equivalent to $\tilde{\beta}$ there exists a homeomorphism $M(\rho, \mu) \to M(\lambda, \tilde{\beta})$ which sends h to the involution h' defined by $[x, t] \to [\lambda(x), t]$ on $M(\lambda, \tilde{\beta})$. Then λ covers a free involution λ^* on S, the orbit space of $\tilde{\beta}$. The map λ^* is equivalent to $\alpha(\varepsilon)$ for $\varepsilon = n_3$, or n_4. Let k^* be a homeomorphism of S such that $k^*\lambda^*k^{*-1} = \alpha(\varepsilon)$. Then k^* lifts to a homeomorphism k of T_{2g-1} such that $k\lambda k^{-1} = \tilde{\alpha}(\varepsilon)$ and k commutes with $\tilde{\beta}$. Define the homeomorphism g: $M(\lambda, \tilde{\beta}) \to M(\tilde{\alpha}(\varepsilon), \tilde{\beta})$ by $g([x, t]) \to [k(x), t]$. Observe that $gh'g^{-1}([x, t]) = [\tilde{\alpha}(\varepsilon)(x), t]$.

All that remains is to show that the involutions $h(\varepsilon)$ on $M(\tilde{\alpha}(\varepsilon), \tilde{\beta})$, defined by $[x, t] \to [\tilde{\alpha}(\varepsilon)(x), t]$, are distinct for $\varepsilon = n_3$ and n_4. First we point out that $\tilde{\beta}$ covers a free involution $\gamma(\varepsilon)$ on $T/\tilde{\alpha}(\varepsilon)$. It is easy to see that the orbit space of $h(\varepsilon)$ is homeomorphic to $M(\gamma(\varepsilon))$. Now the involution $\tilde{\alpha}(\varepsilon)\tilde{\beta}$ covers a homeomorphism σ: $T/\tilde{\beta} \to T/\tilde{\alpha}(\varepsilon)$ such that $\sigma\alpha(\varepsilon)\sigma^{-1} = \gamma(\varepsilon)$. Therefore, by Lemma 8, $h(\varepsilon)$ and $h(\varepsilon')$ have homeomorphic orbit spaces if and only if $\varepsilon = \varepsilon'$.

References

1. E. M. Brown, *Unknotting in* $M^2 \times I$, Trans. Amer. Math. Soc. **123** (1966), 480–505. MR **33** #6640.

2. P. E. Conner and F. Raymond, *Actions of compact Lie groups on aspherical manifolds*, Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969), Markham, Chicago, Ill., 1970, pp. 227–264. MR **42** #6839.

3. W. Heil, *On* P^2*-irreducible 3-manifolds*, Bull. Amer. Math. Soc. **75** (1969), 772–775. MR **40** #4958.

4. ———, *On certain fiberings of* $M^2 \times S^1$, Proc. Amer. Math. Soc. **34** (1972), 280–286. MR **46** #2675.

5. ———, *Almost sufficiently large Seifert fiber spaces*, Michigan Math. J. **20** (1973), 217–223. MR **51** #14060.

6. P. K. Kim, *PL involutions on lens spaces and other 3-manifolds*, Proc. Amer. Math. Soc. **44** (1974), 467–473.

7. P. K. Kim and J. Tollefson, *Splitting the PL involutions on nonprime 3-manifolds*, Michigan Math. J. (to appear).

8. K. W. Kwun, *Scarcity of orientation-reversing* PL *involutions of lens spaces*, Michigan Math. J. **17** (1970), 355–358. MR **43** #5535.

9. K. W. Kwun and J. L. Tollefson, *PL involutions of* $S^1 \times S^1 \times S^1$, Trans. Amer. Math. Soc. **203** (1975), 97–106.

10. ———, *Extending a PL involution of the interior of a compact manifold*, Amer. J. Math. (to appear).

11. G. R. Livesay, *Fixed point free involutions on the 3-sphere*, Ann. of Math. (2) **72** (1960), 603–611. MR **22** #7131.

12. ———, *Involutions with two fixed points on the three-sphere*, Ann. of Math. (2) **78** (1963), 582–593. MR **27** #5257.

13. P. Orlik, *On the existence of the infinite cyclic group by a 2-manifold group*, Illinois J. Math. **12** (1968), 479–482. MR **37** #1441.

14. P. Orlik and F. Raymond, *On 3-manifolds with local* SO(2) action, Quart. J. Math. Oxford Ser. (2) **20** (1969), 143–160. MR **42** #1121.

15. H. Seifert, *Topologie dreidimensionaler gefaserter Räume*, Acta Math. **60** (1933), 147–283.

16. J. Stallings, *On the loop theorem*, Ann. of Math. (2) **72** (1960), 12–19. MR **22** #12526.

17. J. L. Tollefson, *3-manifolds fibering over* S^1 *with nonunique connected fiber*, Proc. Amer. Math. Soc. **21** (1969), 79–80. MR **38** #5238.

18. ———, *Involutions on* $S^1 \times S^2$ *and other 3-manifolds*, Trans. Amer. Math. Soc. **183** (1973), 139–152. MR **48** #5081.

19. ———, *A 3-manifold admitting a unique periodic* PL *map*, Michigan Math. J. **21** (1974), 7–12. MR **50** #3232.

20. F. Waldhausen, *Gruppen mit Zentrum und 3-dimensionale Mannigfaltigkeiten*, Topology **6** (1967), 505–517. MR **38** #5223.

21. ———, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88. MR **36** #7146.

22. ———, *Über Involutionen der 3-Sphäre*, Topology **8** (1969), 81–91. MR **38** #5209.

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